Spatial search with two marked vertices is optimal for almost all queries

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Introduction

Plan:

1. Searching *one* marked element with a quantum walk (literature)

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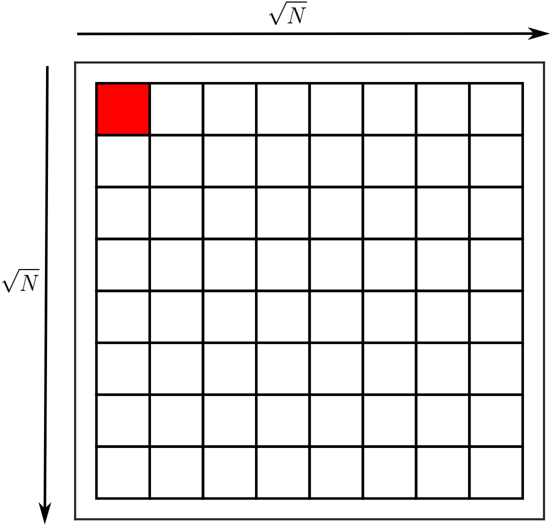
2. Searching *two* marked elements with a quantum walk (analytical analysis)
3. Searching M marked elements with a quantum walk (statistical analysis)

Spatial search on a grid



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where $\alpha \in \{0, 1\}$. The oracle is given by

$$R = \mathbb{I} - 2 \sum_{m \in \mathcal{M}} \sum_{0 \leq i, j \leq 1} \left|i, m\right\rangle \left\langle j, m\right| = \mathbb{I} - 2 \sum_{m \in \mathcal{M}} \left|d, m\right\rangle \left\langle d, m\right|.$$

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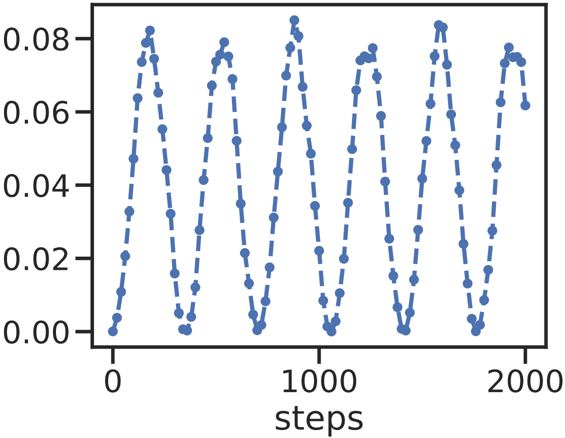
Operator for the searching algorithm:

U'=UR

Signal of a quantum walk search



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Complexity

After repeating a logarithmic number of times, the algorithm succeed with probability $1-\epsilon$ and has a complexity of

$$O\left(\sqrt{N}\ \ln^{3/2} N\ \ln 1/\epsilon\right)$$

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- Do we keep the same complexity ?
- Does this depend of the relative position ?
- How the hitting time is affected ?
- How the probability of success is affected ?

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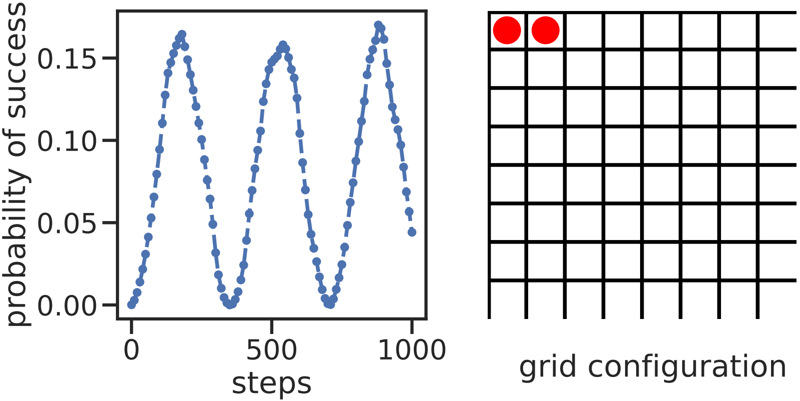
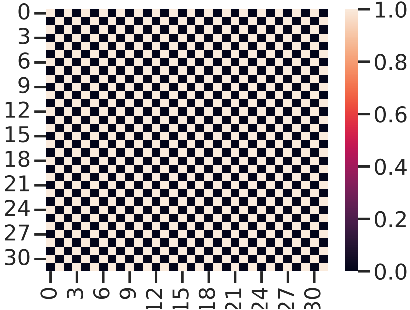
- Do we keep the same complexity ? **Spoiler : Almost always**
- Does this depend of the relative position ? **Spoiler : Yes**
- How the hitting time is affected ?
- How the probability of success is affected ?

Case 1 : Nothing happens



Case 1 : What do we know ?



- Same complexity.

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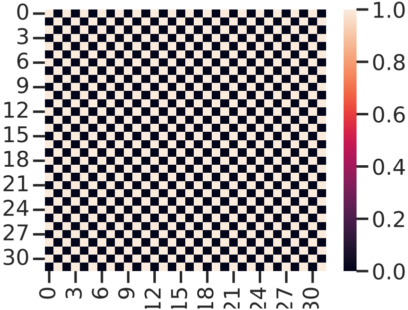

- Same complexity.
- Half of the configuration are concerned.

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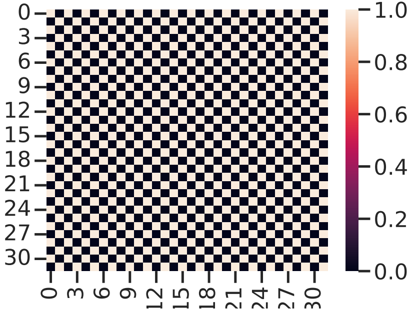

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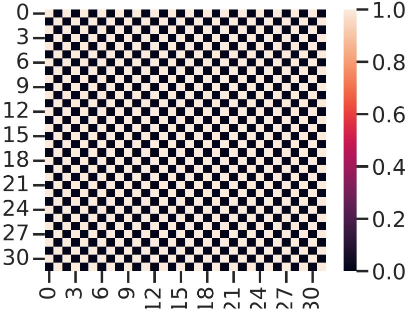

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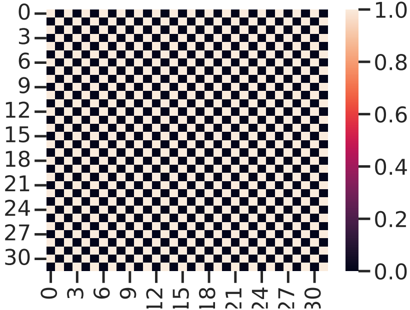

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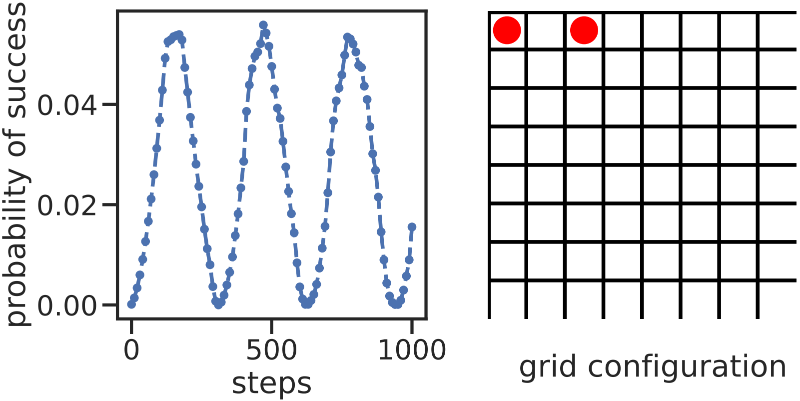
Theorem

While searching two elements $m_0 = (0,0)$ *and* $m_1 = (x,y)$*, if* $x + y$ *is odd then*

$$p_s \sim \frac{\pi}{2 \ln N} \qquad \textit{and} \qquad t_{opt} \sim \frac{\sqrt{\pi N \ln N}}{4}.$$

Case 2 : Some not too bad interferences



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While searching two elements $m_0 = (0,0)$ *and* $m_1 = (0,2)$ *it holds that*

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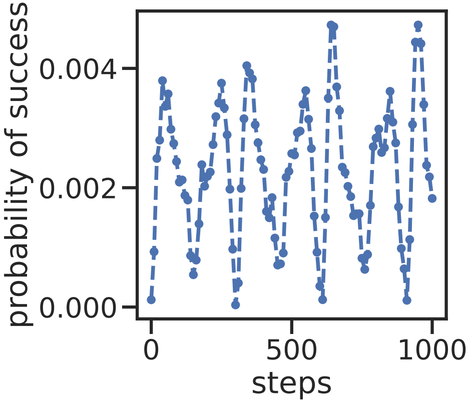
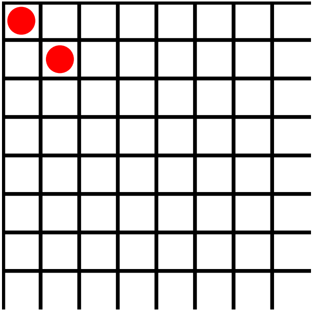
- Same complexity.
- The probability of success and hitting time may change.

Case 3 : We lose optimality





grid configuration

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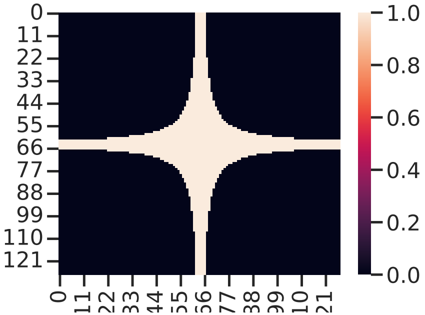

- $p_s = O(1/N)$.

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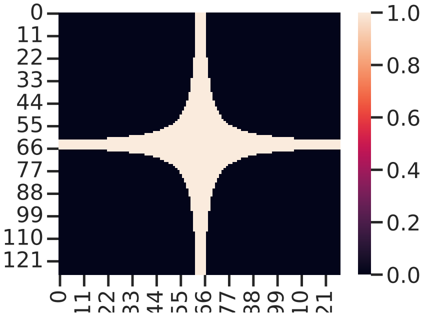

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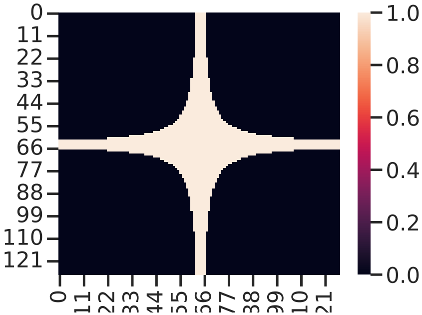

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Theorem

While searching two elements $m_0 = (x_0, y_0)$ and $m_1 = (x_1, y_1)$, a necessary (but not sufficient) condition to have a non-optimal complexity is

$$\left(\overline{x_0 - x_1}\right)^2 \left(\overline{y_0 - y_1}\right)^2 \leq N.$$

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While searching two elements, there is at most $O\left(\sqrt{N \ln N}\right)$ *non-optimal ones.*

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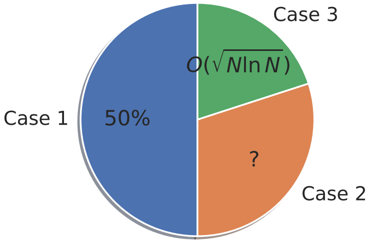

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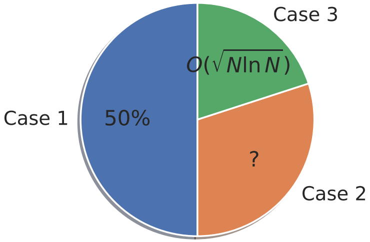

- $N-1$ possible configuration.
- The ratio of non optimal configurations

$$\frac{\mathcal{N}_{\text{non-optimal}}}{N-1} = O\left(\frac{\ln N}{\sqrt{N}}\right) \xrightarrow[N\to\infty]{} 0.$$

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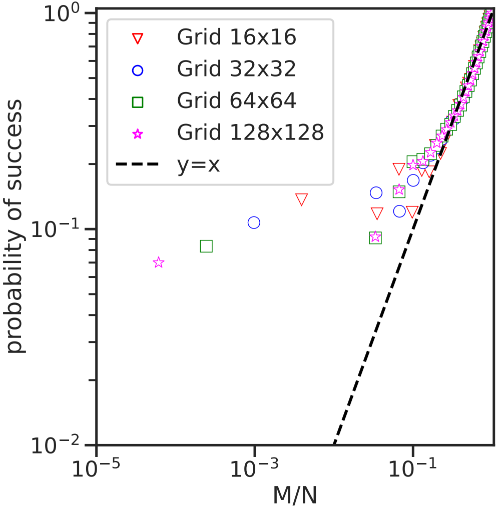
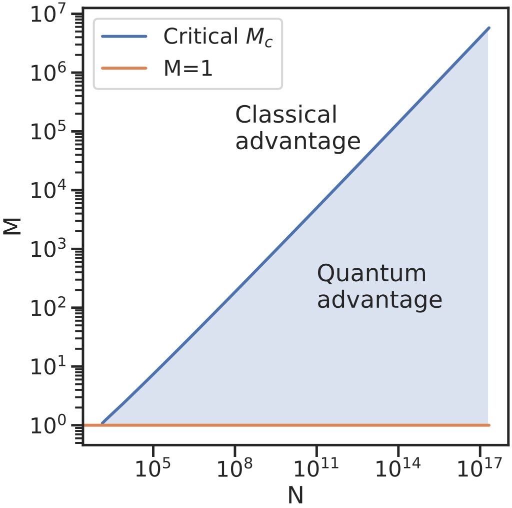
Average probability of success



Do we have a quantum advantage ?



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- For two marked elements, it is possible to have a non optimal configuration.
- For two marked elements, almost all configurations are optimal.
- Statistics suggest that we lose the quantum advantage when the ratio of marked elements increases.